KNOX GRAMMAR SCHOOL

Student Name: ______________________________

Student Number: ______________________________

Teacher's Name: ______________________________

2023 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 1

General Instructions:

- Reading Time – 10 mins
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided on a separate document
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 1–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–12)

- Attempt Questions 11–14
- Allow about 1 hours and 45 minutes for this section

Section I

10 Marks

Attempt Questions 1 – 10 (1 mark for each question)

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-10

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1. Given that $\cos^{-1} a = b$ where b is an obtuse angle, which of the following could the value $\sin^{-1} a$?

A. $\frac{8\pi}{5}$

B. $\frac{3\pi}{5}$

C. $-\frac{3\pi}{5}$

D. $-\frac{\pi}{5}$

2. The polynomial $P(x)$ has a degree 6. When $P(x)$ is divided by another polynomial $Q(x)$ the remainder is $x^3 + 1$.

Which of the following is true about the degree of the polynomial $H(x) = \left(P(x)Q(x)\right)^2$?

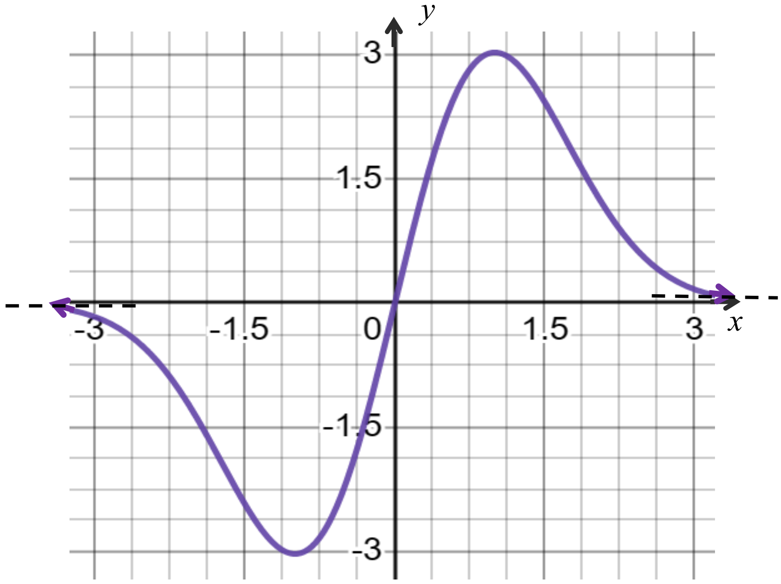
A. The degree could be 10, 11 or 12.

B. The degree could be 18, 20, 22 or 24

C. The degree could be 20, 22 or 24.

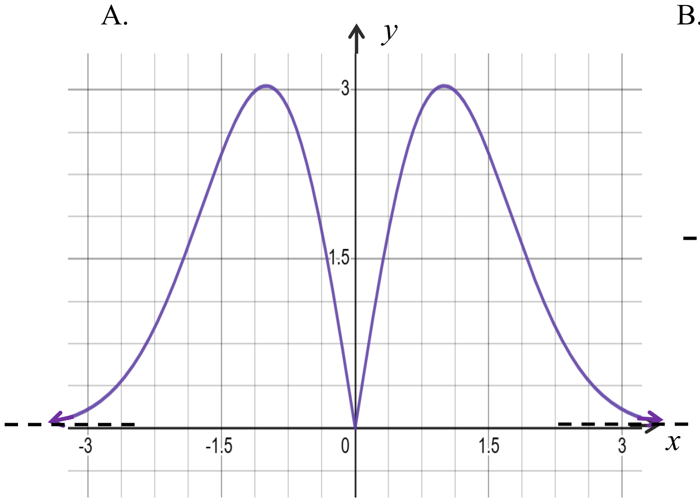
D. The degree could be 22, 24, 26.

3. The diagram shows $y = f(x)$.

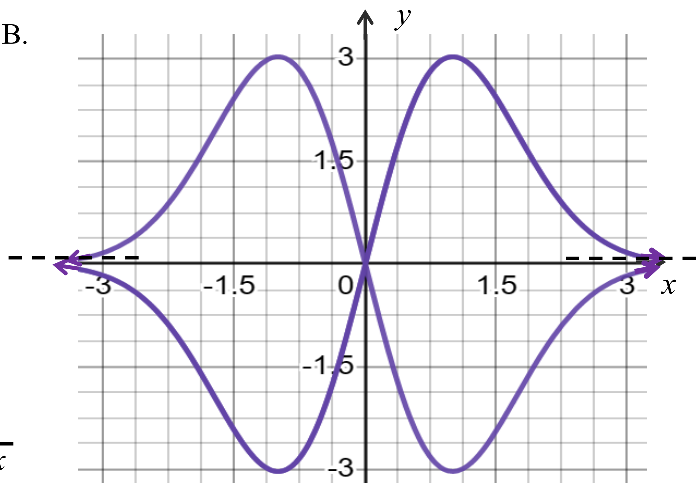


Which of the following is the graph of $|y| = f(|x|)$?

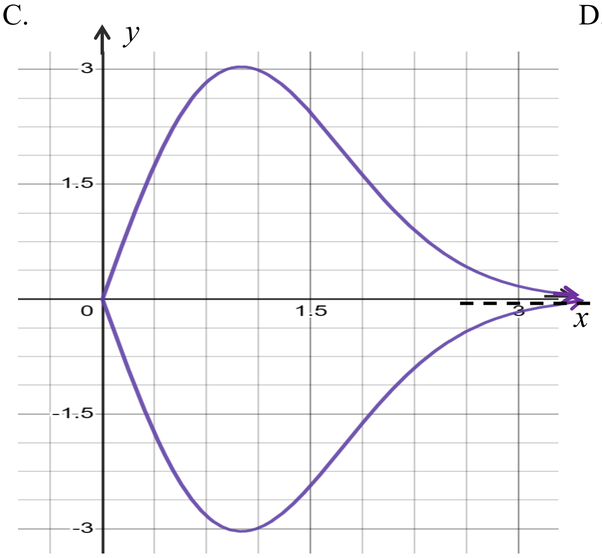
A.



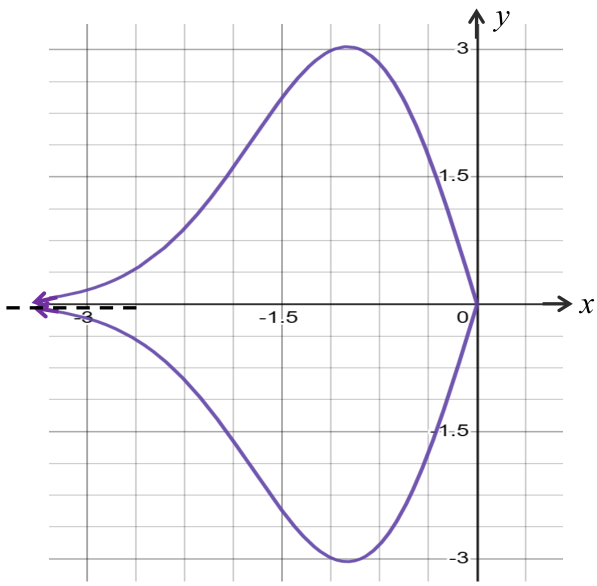
B.



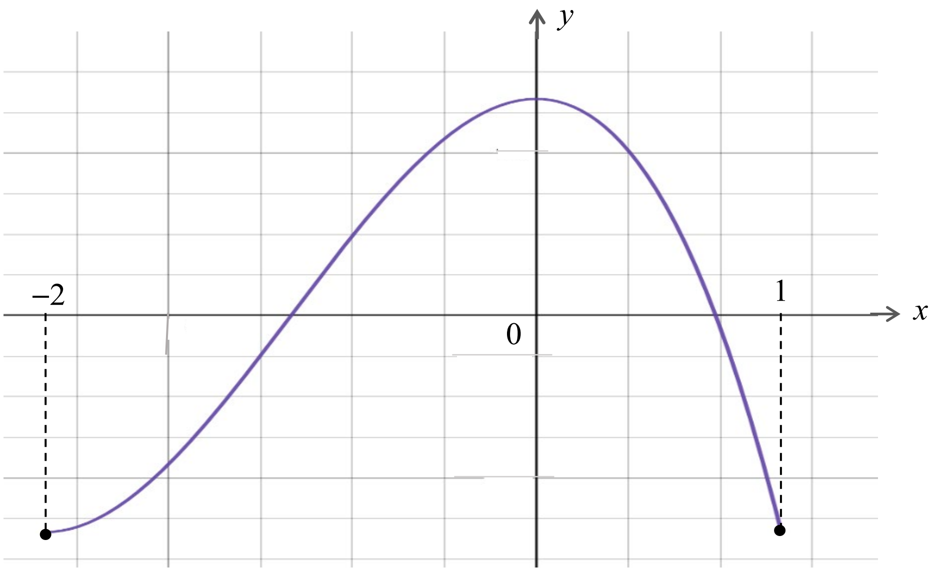
C.



D.



4. The diagram shows the graph of a function.



Which pair of parametric equations represents this function ?

A. $x = t - 1$ and $y = 3t - t^3$, where $-1 \le t \le 2$.

B. $x = t + 1$ and $y = 3t - t^3$, where $-1 \le t \le 2$.

C. $x = t - 1$ and $y = -3t + t^3$, where $-1 \le t \le 2$.

D. $x = t - 1$ and $y = 3t + t^3$, where $-1 \le t \le 2$.

5. The numbers 2, 4, 5, 6, 8, 10, 12, 16, 40 are written on separate cards and placed in a bag.

What is the smallest number of cards we need to take from the bag to make sure we get at least one pair of numbers that have a product of 80?

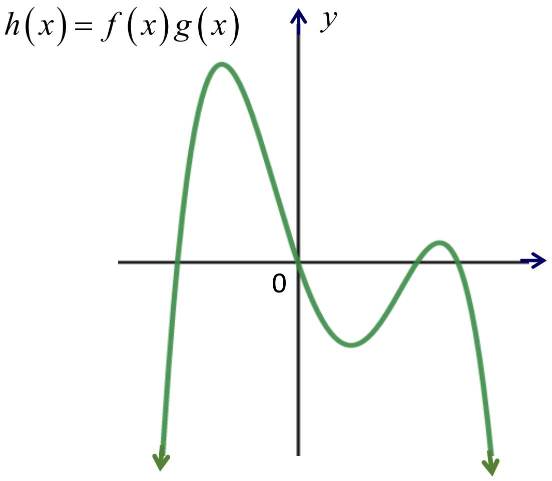
A. 6

B. 7

C. 4

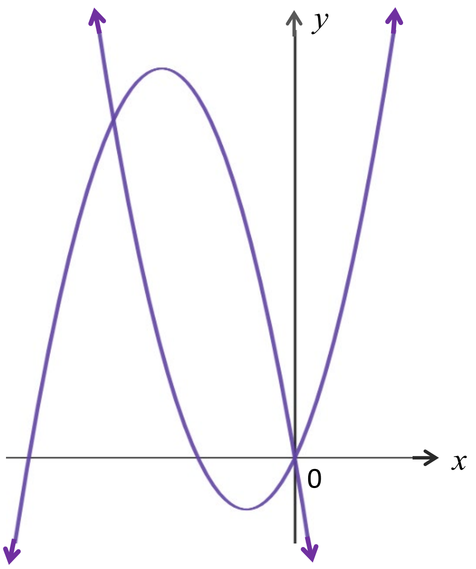
D. 5

6. The diagram shows the graph of $h(x)$ which is the product of the functions $f(x)$ and $g(x)$.

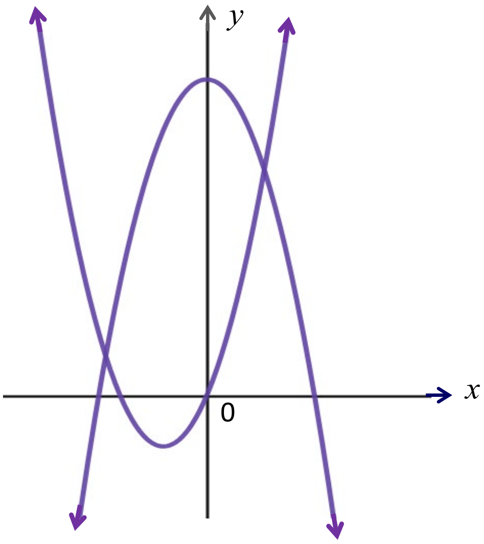


Which of the following is the best option to represent the graphs of two functions $f(x)$ and $g(x)$?

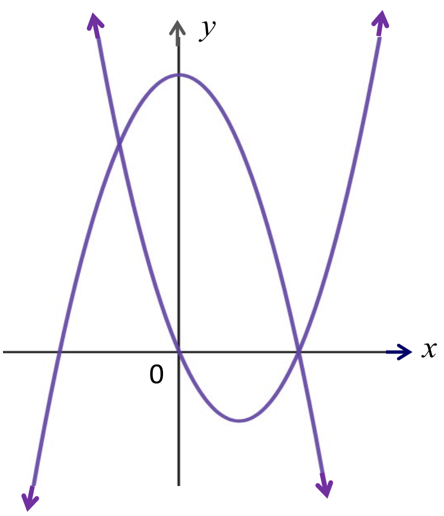
A.



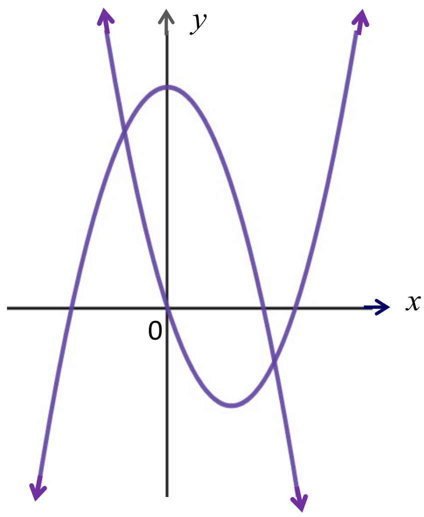
B.



C.

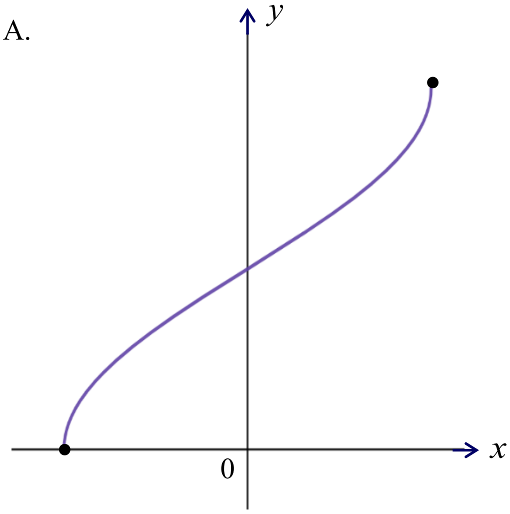


D.

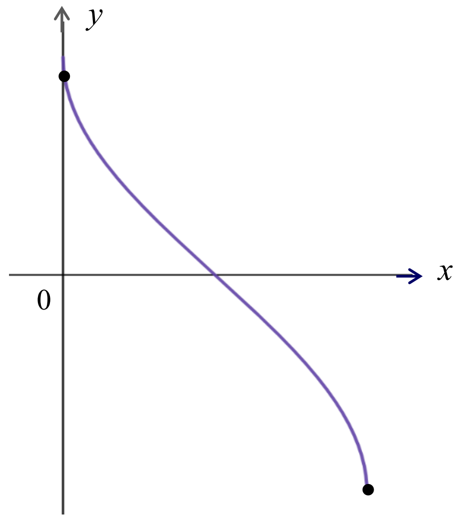


7. Which of the following options could be part of the graphical solution to the differential equation $\dfrac{dy}{dx} = \dfrac{1}{4\sin y \cos y}$?

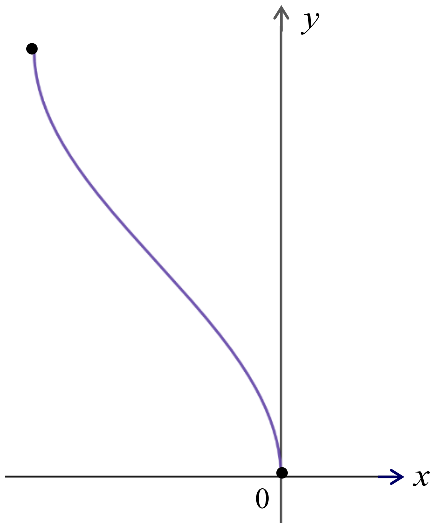
A.



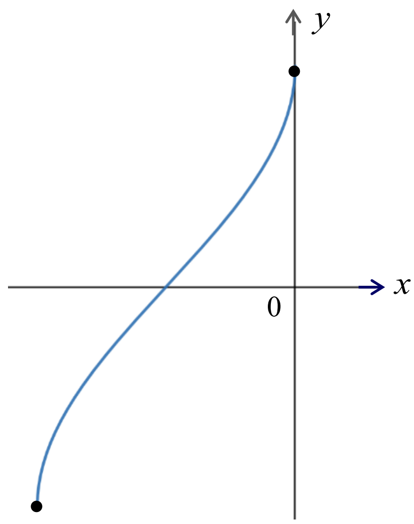
B.



C.



D.



8. The length of the projection of vector $\underset{\sim}{p} = 2(\ln x)^2 \underset{\sim}{i} + 3\ln x \underset{\sim}{j}$, where $\ln x$ is a positive integer, on to $\underset{\sim}{q} = -5\underset{\sim}{i} + 12\underset{\sim}{j}$ is $\dfrac{32}{13}$. What is the value of $\left|\underset{\sim}{p}\right|$?

A. 5

B. 10

C. 13

D. 15

9. The magnitudes of two vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are 3 and 2 respectively.

The angle between these two vectors is θ such that $\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$.

Which of the following is the correct range of values for $\left|\underset{\sim}{p} - \underset{\sim}{q}\right|$?

A. $7 \leq \left|\underset{\sim}{p} - \underset{\sim}{q}\right| \leq 19$

B. $7 \leq \left|\underset{\sim}{p} - \underset{\sim}{q}\right| \leq 13$

C. $\sqrt{7} \leq \left|\underset{\sim}{p} - \underset{\sim}{q}\right| \leq \sqrt{19}$

D. $\sqrt{7} \leq \left|\underset{\sim}{p} - \underset{\sim}{q}\right| \leq 13$

10 . A one-to-one function $g(x)$ has domain $x \geq 0$ and range $y \geq 0$.

$g(x)$ passes through the origin and its inverse function is $g^{-1}(x)$.

Given that $\int_0^a \left(g^{-1}(x) - x\right)dx = \frac{3}{2}a^2$ and $g^{-1}(x) \neq x$ for $0 \leq x \leq a$,

what is the value of $\int_0^{g^{-1}(a)} g(x)\, dx$?

A. $ag^{-1}(a) - 2a^2$

B. $ag^{-1}(a) + 2a^2$

C. $ag^{-1}(a) - \frac{3}{2}a^2$

D. $ag^{-1}(a) + \frac{3}{2}a^2$

Section II

60 marks

Attempt Questions 11 – 14

Allow about 1 hour and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new Writing Booklet.

(a) Find the coefficient of x^4 in the expansion of $\left(x+\frac{2}{x}\right)^{10}$. **2**

(b) Solve the inequality $\frac{1}{x-3} \geq -1$. **3**

(c) The polynomial $P(x)=3x^3-ax^2-8x+b$ has $x=-2$ as a root of multiplicity 2. **2**

Find the values of a and b.

(d) Use the substitution $u=\tan x-x$ to find $\int \tan^2 x\sqrt{\tan x-x}\,dx$. **3**

(e) In a large city, the probability that a person has red hair is 0.18. **2**

Let $\hat{p}$ be the random variable that represents the sample proportion of red haired people from a sample of size n, drawn from the city.

Assuming that sample proportion is approximately normally distributed, what is the smallest value of n such that the standard deviation of $\hat{p}$ is less than or equal to 0.07?

(f) Consider the vectors $\underset{\sim}{a}=\underset{\sim}{i}-2\underset{\sim}{j}$ and $\underset{\sim}{b}=x\underset{\sim}{i}+y\underset{\sim}{j}$ where x and y are real. **3**

It is known that $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$ and $\left|\underset{\sim}{b}\right|=2\sqrt{5}$.

Find the possible values of x and y.

End of Question 11

Question 12 (15 marks) Start a new Writing Booklet.

(a) Consider the differential equation $\dfrac{dy}{dx} = \dfrac{2x+y}{x^3-y^3}$. **2**

Draw the correct slopes of the direction field at the points $A(0,-2)$, $B(-1,2)$ and $C(1,1)$.

(b) A firework is projected from the point $A(0, 10)$ with initial velocity of magnitude V m s^{-1} at an angle θ to the horizontal.

Before reaching its maximum height, the firework explodes at a point B.
The time taken to reach B was 2 seconds.

Its velocity at B is 34 ms^{-1} where its path makes an angle α with the horizontal such that $\tan\alpha = \dfrac{8}{15}$.

The vector position of the firework is given by $\underset{\sim}{r} = (Vt\cos\theta)\underset{\sim}{i} + (10 + Vt\sin\theta - 5t^2)\underset{\sim}{j}$.

(i) Show that $V\sin\theta = 36$. **3**

(ii) Find V, the magnitude of the initial velocity, and the size of the angle of projection θ. **2**

(c) The gradient of the tangent to the curve $g(x) = \left(\sin^{-1}\dfrac{x}{b}\right)^2$ at the point A with x coordinate $\dfrac{b}{2}$ is $\dfrac{2\pi}{3}$. **3**

Find the value of b.

(d) There are n dancers in a performing arts academy. **2**

The trainer is to select a group of two or three of these dancers to perform in a certain concert.

The number of different groups of three dancers he can select is 3 times the number of different groups of two dancers.

Find n the number dancers in the performing arts academy.

(e) Use mathematical induction to prove that, for any integer $n \geq 1$, **3**

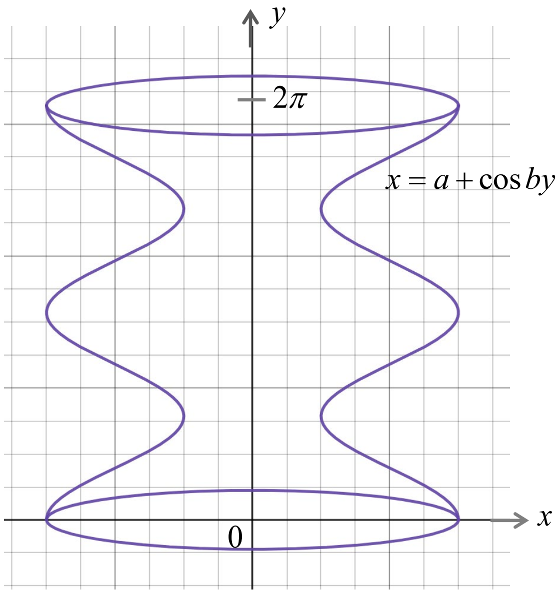
$9^{n+3} - 2^{2n+2}$ is divisible by 5.

End of Question 12

Question 13 (15 marks) Start a new Writing Booklet.

(a) Consider the part of the curve with equation $x = a + \cos by$, where $0 \leq y \leq 2\pi$ and $a > 0$ and b are constants.

A vase is made by rotating this part of the curve about the y-axis as shown below.



NOT TO SCALE

(i) Find the value of b. **1**

(ii) Given that the exact volume of the vase is $19\pi^2$ cubic units, find the value of a. **2**

(b) A juice company sells 375 mL juice bottles. **3**

The product manager of the juice company claims that "75% of their juice bottles contain more than 375mL".

During a routine check the director of the factory checked 18 bottles and found 9 of them contained less than 375mL.

Assuming that the claim of the product manager is correct, the director used the Normal approximation to the binomial distribution to calculate for a random sample of 18 the probability P of having at least 9 bottles containing less than 375mL.

Find the value the value of P and explain why this method might not be valid.

You may use the information on page 13 to answer this question.

Question 13 continues on page 10

(c) Consider the monic polynomial $g(x)$ with degree 3 and roots α, β and γ. **3**

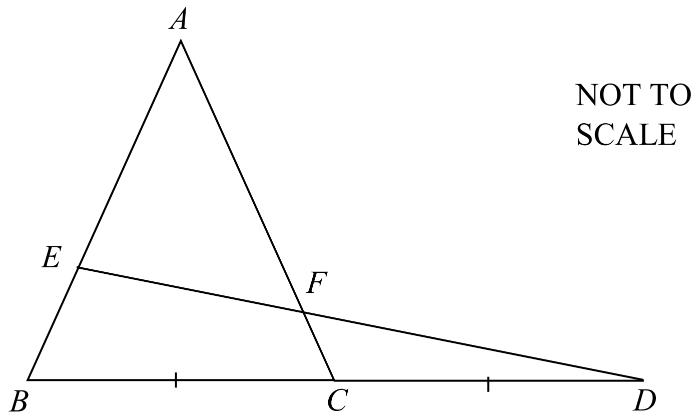
It is given that $g''(\alpha+\beta+\gamma)=20$ and $g''(\alpha^2+\beta^2+\gamma^2)=164$.

Find the value of $\alpha(\beta+\gamma)+\beta(\alpha+\gamma)+\gamma(\alpha+\beta)$.

(d) The diagram shows an isosceles triangle *ABC* with *AB* = *AC*. **3**
The base *BC* is produced to *D* such that *CD* = *BC*.

E and F are points on AB and AC respectively such that

$\overrightarrow{AE}=(k+1)\underset{\sim}{u}$, $\overrightarrow{EB}=2\underset{\sim}{u}$, $\overrightarrow{AF}=(2k-1)\underset{\sim}{v}$ and $\overrightarrow{FC}=\underset{\sim}{v}$.



Given that $\overrightarrow{EF}=p\,\overrightarrow{ED}$, find the value of the scalar quantity p.

(e) The function $f(x)$ is defined in the domain $x \geq e$ and its derivative is **3**

$f'(x)=x\ln x - x + 1$.

Show that $f(x)$ has an inverse function $f^{-1}(x)$ in the domain $x \geq e$.

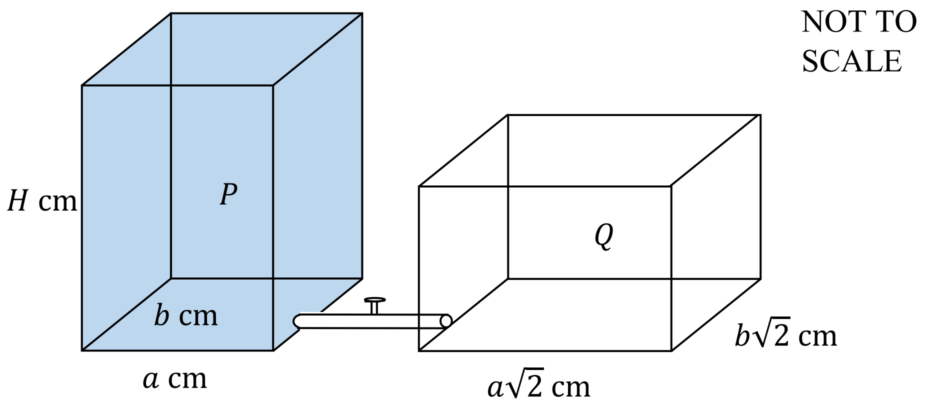
End of Question 13

Question 14 (15 marks) Start a new Writing Booklet.

(a) Consider the differential equation $\left(e^x + 1\right)\dfrac{dy}{dx} = e^x \sec y$. **3**

Find the particular solution of this equation given that it passes through the origin.

(b) The diagram shows two tanks P and Q connected by a small valve.



Initially, tank P is full of liquid and tank Q is empty.

At time $t = 0$, the small valve opens and the liquid from tank P passes into tank Q.

The height of liquid in tank Q increases at a rate proportional to the difference in the heights of liquid in the two tanks.

At a time t, the level of liquid in tank Q is h cm.

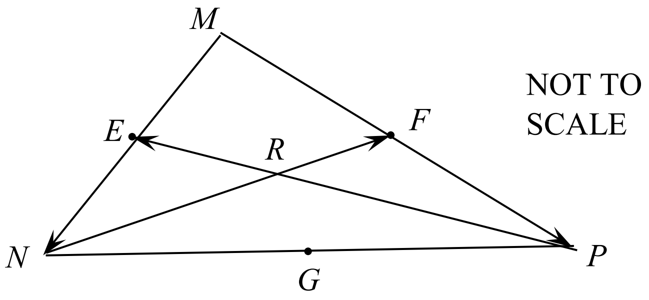
(i) Show that $\dfrac{dh}{dt} = k\left(H - 3h\right)$, where k is a positive constant. **2**

(ii) By solving the given differential equation show that $h = \dfrac{H}{3}\left(1 - e^{-3kt}\right)$. **2**

(iii) After 4 seconds, the level of liquid in tank Q is h_1 cm and after 12 seconds is h_2 cm. Given that $h_2 = \dfrac{7}{4}h_1$, find in terms of H the level of the liquid in tank Q after 16 seconds. **4**

Question 14 continues on page 12

(c) In the triangle MNP shown in the diagram the points E, F and G are the midpoints of MN, MP and NP respectively. M **4**

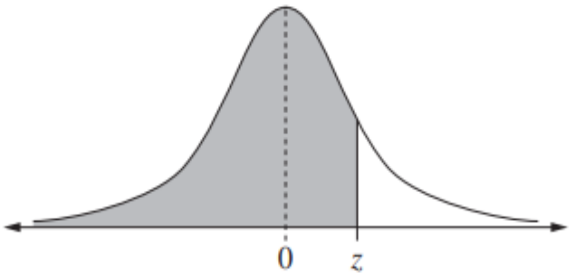


Let $\overrightarrow{MN} = \underset{\sim}{n}$, $\overrightarrow{MP} = \underset{\sim}{p}$, $\overrightarrow{NR} = x\,\overrightarrow{NF}$ and $\overrightarrow{PR} = y\,\overrightarrow{PE}$, where x and y are real numbers between 0 and 1.

By finding two expressions for $\overrightarrow{MR}$, show that the medians $\overrightarrow{PE}$, $\overrightarrow{NF}$ and $\overrightarrow{MG}$ are concurrent and that R divides each median in the ratio 2:1.

End of paper

Table of values $P(Z \leq z)$ for the normal distribution $N(0,1)$

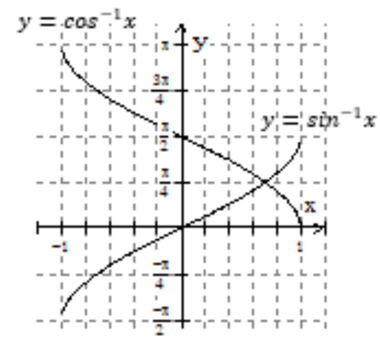


Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

2023 Year 12 Mathematics X1 Trial Examination SOLUTIONS

Suggested Solution (s)	Comments

1. **D**



As b is an obtuse angle then $-1 < a < 0$ and from the graph we can see that for this range of values of a that $-\frac{\pi}{2} < sin^{-1}a < 0$.
This indicates that the only valid option is $-\frac{\pi}{5}$.
Hence, the correct option is **D**.

2. **C**

When $P(x)$ is divided by $Q(x)$ the remainder is $x^3 + 1$. As $P(x)$ has degree 6, and the remainder has degree 3, the polynomial $Q(x)$ must have degree 4, 5 or 6. This means $P(x) \times Q(x)$ must have degree 10, 11 or 12. Hence $\left(P(x)\,Q(x)\right)^2$ must have degree 20, 22 or 24.
Hence, the correct option is **C**.

3. **B**

The graph of $y = f(|x|)$ keeps the part of the graph which is to the right of the y axis and adds its reflection across the y axis to form the final graph.
The graph of $|y| = f(x)$ keeps the part of the graph above the x axis and adds its reflection across the x axis to form the final graph.
So, the graph of $|y| = f(|x|)$ involves both of these transformations, which taken in either order will produce the graph shown in option **B**.

4. **A**

In all options we have $-1 \le t \le 2$ this means $-2 \le t - 1 \le 1$ and as from the graph $-2 \le x \le 1$ we can easily deduce that $x = t - 1$.
This indicates that option B is invalid.

Using the table below we can see that option C is invalid as $(-2, 2)$ is not a point on the graph.

	t	x	y
Option C	-1	-2	2

Also, using the following table below we can see that option D is invalid as $(-2, 2)$ is not a point on the graph.

	t	x	y
Option D	2	1	4

Now, Options B , C and D are invalid then option must be valid. In addition, if we substitute any value of t such that $-1 \le t \le 2$ we obtain a point on the curve.
Hence, the correct option is **A**.

5. **B**

The pairs of numbers which produce a product of 80 are 2 × 40, 5 x 16, and 8 x 10. To find how many numbers we must select before we can guarantee a pair that will produce a product of 80, we must examine the worst possible case. The first 3 numbers selected could be 4, 6 and 12 which cannot form a pair with a product of 80. The next 3 could be one from each of the pairs which could form a product of 80. The next number selected must be the other half of one of the 3 pairs that will produce a product of 80. So, the smallest number of cards to be selected before we get at least one pair of numbers with a product of 80 is 7.
Hence, the correct option is **B**.

6. **D**

As $h(x)$ has 4 zeros, the graphs of $f(x)$ and $g(x)$ must have a total of 4 distinct zeros, which means the correct option must be B or D. Option B has 2 of its zeros on the negative part of the x axis, but $h(x)$ has only one zero on the negative part of the x axis, another zero at the origin and 2 zeros on the positive part of the x axis, which matches the zeros of the 2 graphs in option D exactly.
Hence, the correct option is **D**.

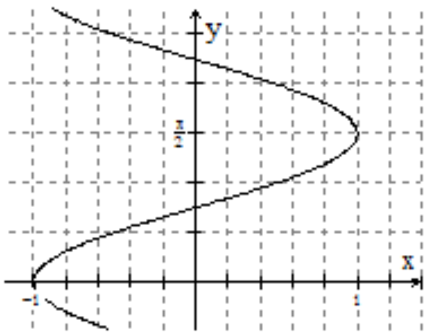
7. **A**

7. **A**

As $\frac{dy}{dx} = \frac{1}{4\sin y \cos y}$ *then* $\frac{dx}{dy} = 4\sin y \cos y$

$= 2\sin 2y$

So $x = -\cos 2y$ as shown below.



Hence, the correct option is **A**.

8. **B**

The length of the vector projection of vector $\underset{\sim}{p}$, onto vector $\underset{\sim}{q}$ is $\frac{\underset{\sim}{p}\cdot\underset{\sim}{q}}{|\underset{\sim}{q}|}$ which we are given as $\frac{32}{13}$.

Now, $|\underset{\sim}{q}| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13$

and $\underset{\sim}{p}\cdot\underset{\sim}{q} = -10\ln^2 x + 36\ln x$ this indicates that

$\frac{\underset{\sim}{p}\cdot\underset{\sim}{q}}{|\underset{\sim}{q}|} = \frac{-10\ln^2 x + 36\ln x}{13} = \frac{32}{13}$ that is

$-10(\ln x)^2 + 36\ln x = 32$ that is

$10(\ln x)^2 + 36\ln x + 32 = 0$

$10m^2 - 36m + 32 = 0$

$2(5m - 8)(m - 2) = 0$ that is $m = 2$ *or* $m = \frac{8}{5}$.

As $m = \ln x$ is a positive integer then only $\ln x = 2$ is valid.

Therefore, $\underset{\sim}{p} = 8\underset{\sim}{i} + 6\underset{\sim}{j}$ so $|\underset{\sim}{p}| = \sqrt{8^2 + 6^2} = 10$.

Hence, the correct option is **B**.

9. **C**

Using the cosine rule

$|\underset{\sim}{p} - \underset{\sim}{q}|^2 = |\underset{\sim}{p}|^2 + |\underset{\sim}{q}|^2 - 2|\underset{\sim}{p}||\underset{\sim}{q}|\cos\theta$

When $\theta = \frac{\pi}{3}$, $|\underset{\sim}{p} - \underset{\sim}{q}|^2 = 9 + 4 - 2 \times 3 \times 2 \times \frac{1}{2}$

$|\underset{\sim}{p} - \underset{\sim}{q}| = \sqrt{13 - 6} = \sqrt{7}$

When $\theta = \frac{2\pi}{3}$, $|\underset{\sim}{p} - \underset{\sim}{q}|^2 = 9 + 4 - 2 \times 3 \times 2 \times \left(-\frac{1}{2}\right)$

$|\underset{\sim}{p} - \underset{\sim}{q}| = \sqrt{13 + 6} = \sqrt{19}$

So, $\sqrt{7} \le |\underset{\sim}{p} - \underset{\sim}{q}| \le \sqrt{19}$

Hence, the correct option is **C**.

Alternative method

Using the cosine rule

$|\underset{\sim}{p} - \underset{\sim}{q}|^2 = |\underset{\sim}{p}|^2 + |\underset{\sim}{q}|^2 - 2|\underset{\sim}{p}||\underset{\sim}{q}|\cos\theta$

$= 9 + 4 - 2 \times 3 \times 2\cos\theta$

$= 13 - 12\cos\theta$

Now, we are given $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$ this indicates

$-\frac{1}{2} \le \cos\theta \le \frac{1}{2}$ so $6 \ge -12\cos\theta \ge -6$

Therefore $19 \ge 13 - 12\cos\theta \ge 7$ that is

$19 \ge |\underset{\sim}{p} - \underset{\sim}{q}|^2 \ge 7$.

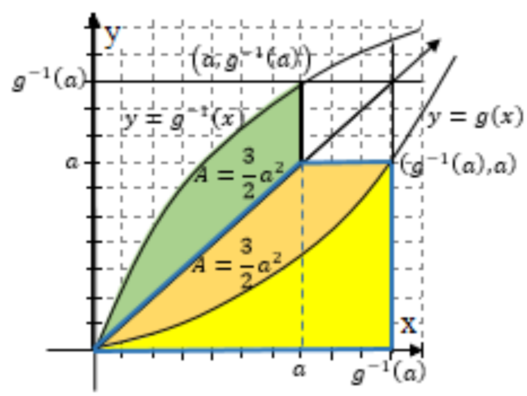
So, $\sqrt{19} \ge |\underset{\sim}{p} - \underset{\sim}{q}| \ge \sqrt{7}$

Hence, the correct option is **C**.

10. **A**

As $g(x)$ is a one to one function, it contains no turning points. Also, as $\int_0^a \left(g^{-1}(x) - x\right) dx = \frac{3}{2}a^2$ which is positive then $g^{-1}(x)$ must be above the line $y = x$ in the domain $0 \le x \le a$.

From the above we can create the graph shown below where the given area which is $\frac{3}{2}a^2$ is coloured green.



Now, $\int_0^{g^{-1}(a)} g(x)\ dx$ is equivalent to the area inside the trapezium minus the mustard yellow area between the curve, the line $y = x$ and the line $y = a$ which is by symmetry equal to $\frac{3}{2}a^2$.

The area of the trapezium is equal to the area of the triangle $A = \frac{1}{2} \times a \times a = \frac{1}{2}a^2$ plus the area of the rectangle $A = a(g^{-1}(a) - a) = a\left(g^{-1}(a)\right) - a^2$

Hence, the required area is

$$\int_0^{g^{-1}(a)} g(x)\ dx = \frac{1}{2}a^2 + ag^{-1}(a) - a^2 - \frac{3}{2}a^2$$

$$= ag^{-1}(a) - 2a^2.$$

Hence, the correct option is **A**.

2023 Year 12 Mathematics X1 Trial Examination SOLUTIONS

QUESTION 11

a) The general term in the expansion of $\left(x+\frac{2}{x}\right)^{10}$ is

$T_{r+1}=\binom{10}{r}x^{10-r}\,(2x^{-1})^r=\binom{10}{r}2^r\,x^{10-r}\,x^{-r}$

So, $T_{r+1}=\binom{10}{r}2^r\,x^{10-2r}$ ✓

Now, we need coefficient of x^4 this means $10-2r=4$ that is $r=3$.

So, the required term is

$T_4=\binom{10}{3}2^3x^{10-6}=120\times 8\times x^4$

Hence, the coefficient of x^4 is 960. ✓

b) To solve $\frac{1}{x-3}\geq -1$, we multiply both sides by the positive factor $(x-3)^2$ and as it is positive, the inequality holds, we get

$(x-3)^2\times\frac{1}{x-3}\geq -(x-3)^2$, where $x-3\neq 0$ ✓

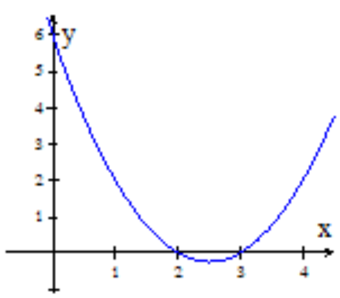
$x-3\geq -(x-3)^2$

$(x-3)+(x-3)^2\geq 0$

$(x-3)\,[1+(x-3)]\geq 0$

$(x-3)(x-2)\geq 0$ ✓

First, we graph $y=(x-3)(x-2)$.



Now, by considering the parts of this graph, at which the y values are positive or above the x axis we get that the solution of $\frac{1}{x-3}\geq -1$ which is $x\leq 2$ or $x>3$ as $x\neq 3$. ✓

Alternative method

$\frac{1}{x-3}\geq -1$ which means $\frac{1}{x-3}+1\geq 0$.

that is $\frac{1}{x-3}+\frac{x-3}{x-3}\geq 0$ that is $\frac{x-2}{x-3}\geq 0$.

x		2		3	
$x-2$	−	0	+	+	+
$x-3$	−	−	−	undefine	+
Solutions	+	0	−	undefine	+

From the table we can see that the solution is $x\leq 2$ or $x>3$ as $x\neq 3$.

c) $P(x)=3x^3-ax^2-8x+b$

As $x=-2$ is a root of multiplicity 2 of $P(x)$ then the three roots can be written as -2, -2 and α.

Using the sum of the roots two at a time, we get

$4-2\alpha-2\alpha=-\frac{8}{3}$ that is

$-4\alpha=-\frac{8}{3}-4$. Multiplying by 3, we get

$-12\alpha=-20$ So, $\alpha=\frac{5}{3}$.

Using the product of the roots, we get

$-2\times -2\times\frac{5}{3}=-\frac{b}{3}$ this means

$\frac{20}{3}=-\frac{b}{3}$ So, $b=-20$. ✓

Using the sum of the roots, we get

$-2-2+\frac{5}{3}=\frac{a}{3}$ this means $-\frac{7}{3}=\frac{a}{3}$

So, $a=-7$. ✓

d) $I=\int\tan^2 x\sqrt{\tan x-x}\;dx$

Let $u=\tan x-x$

so $du=(sec^2x-1)dx$ that is $du=tan^2x\,dx$ ✓

So $I=\int u^{\frac{1}{2}}du=\frac{2}{3}u^{\frac{3}{2}}+c$ ✓

$=\frac{2}{3}(\tan x-x)^{\frac{3}{2}}+c$ ✓

e) Standard Deviation of a sample proportion which is approximately normally distributed is

$\sigma=\sqrt{\frac{pq}{n}}$. As $\sigma\leq 0.07$ then $\sqrt{\frac{0.18\times 0.82}{n}}\leq 0.07$ ✓

Also, as both sides are positive, we can square both sides, we get $\frac{0.1476}{n}\leq 0.0049$.

Now, as $n\geq 0$ we can multiply both sides by n, we get $0.1476\leq 0.0049\,n$ this means $n\geq 30.12244..$

As n must be a positive integer, the smallest possible value of n will be 31. ✓

f) As $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$, then

$\underset{\sim}{a}\cdot\underset{\sim}{b}=x-2y=0$ that is $x=2y$ (1)

Also, $|\underset{\sim}{b}|=\sqrt{x^2+y^2}=2\sqrt{5}$.

By squaring both sides, we get

$x^2+y^2=20$ (2) ✓

By substituting (1) in (2), we get ✓

$4y^2+y^2=20$ this means $5y^2=20$ that is $y^2=4$

So, $y=\pm 2$.

Now, when $y=2, x=4$ and $y=-2, x=-4$

Hence, the possible values of x and y are

$x=4, y=2$ or $x=-4, y=-2$. ✓

QUESTION 12

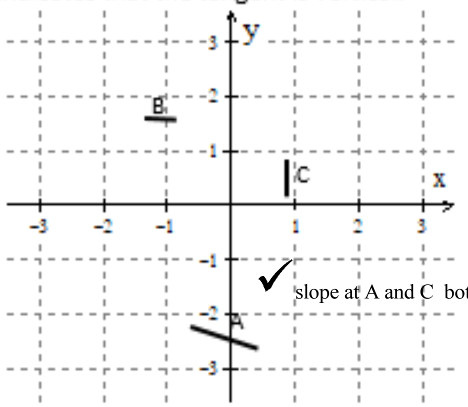
a) At $A(0,-2)$, $\frac{dy}{dx} = \frac{2\times 0\ -2}{0^3-(-2)^3} = \frac{-2}{8} = -\frac{1}{4}$.

At $B(-1,2)$, $\frac{dy}{dx} = \frac{2\times -1+2}{(-1)^3-2^3} = 0$. ✓ slope at B correct

This indicates that the tangent is horizontal.

At $C(1,1)$, $\frac{dy}{dx} = \frac{2\times 1+1}{1^3-1^3} = \frac{3}{0}$ $(undefined)$.

This indicates that the tangent is vertical.



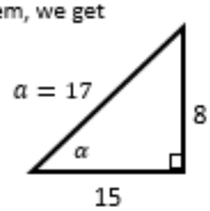
✓ slope at A and C both correct

b) i) We are given that the velocity vector at B is 34 m s^{-1} and its direction is $\tan\alpha = \frac{8}{15}$. Now, to find the vertical and horizontal components of the velocity at B we need to find first $sin\ \alpha$ and $cos\ \alpha$.

Using Pythagoras theorem, we get

$a = \sqrt{8^2+15^2} = 17$.

This means the vertical and horizontal components of the velocity at B are



$\frac{dy}{dt} = 34\ sin\ \alpha$ and $\frac{dx}{dt} = 34\ cos\ \alpha$

$\frac{dy}{dt} = 34\times\frac{8}{17}$ $\qquad \frac{dx}{dt} = 34\times\frac{15}{17}$

$\frac{dy}{dt} = 16\ \text{m}s^{-1}$ (1) $\qquad \frac{dx}{dt} = 30\ \text{m}s^{-1}$ (2) ✓ either 1 or 2

Now, we are given $y = 10 + \text{V}\,t\sin\theta - 5t^2$

this means the vertical component of the velocity is

$\frac{\text{dy}}{\text{dt}} = \text{V}\sin\theta - 10t$ ✓

As the vertical component velocity at B is 16 $\text{m}s^{-1}$ from (1) and the time to reach B is $t = 2$ so by substitution, we get $16 = V\sin\theta - 10\times 2$ this means $V\sin\theta - 20 = 16$ Hence, $V\sin\theta = 36$ (3) ✓ correctly shown

ii) The horizontal component of the velocity is constant. This means the horizontal component of the velocity at B which is 30 $\text{m}s^{-1}$ from (2) equals the horizontal component of velocity at A which is $V\cos\theta$.

Hence, $V\cos\theta = 30$ (4)

Dividing (3) by (4), we get

$\frac{V\sin\theta}{V\cos\theta} = \frac{36}{30}$ that is $\tan\theta = \frac{6}{5}$

Hence, $\theta \approx 50°12'$. ✓

By squaring (4) and (5) then adding the results,

we get $V^2\sin^2\theta + V^2cos^2\theta = 36^2 + 30^2$

$\text{V}^2(\sin^2\theta + \cos^2\theta) = 1296 + 900$

$V^2 = 2\,196$ So, $V = 6\sqrt{61} = 46.86$ ($2d.p$) as $V > 0$. ✓

Hence, the initial velocity of the firework is 46.86 m s^{-1} and the angle of projection is approximately $50°12'$.

c) $g(x) = \left(\text{arcsin}\,\frac{x}{b}\right)^2$ as $\text{arcsin}\,\frac{x}{b} = sin^{-1}\frac{x}{b}$

then $g(x) = \left(sin^{-1}\,\frac{x}{b}\right)^2$.

$g'(x) = \frac{d}{dx}\left(sin^{-1}\frac{x}{b}\right)^2$

$= 2\times sin^{-1}\frac{x}{b}\times\frac{d}{dx}\left(sin^{-1}\frac{x}{b}\right)$ (1)

Note: $\frac{\text{d}}{\text{dx}}\left(sin^{-1}\frac{x}{b}\right) = \frac{\frac{1}{b}}{\sqrt{1-\frac{x^2}{b^2}}}$ ✓

$= \frac{1}{b\sqrt{1-\frac{x^2}{b^2}}} = \frac{1}{\sqrt{b^2-x^2}}$ (2)

By substituting (2) in (1), we get

$g'(x) = 2\times sin^{-1}\frac{x}{b}\times\frac{1}{\sqrt{b^2-x^2}}$

Now, the gradient of the tangent is

$g'\left(\frac{b}{2}\right) = 2\times sin^{-1}\left(\frac{1}{b}\times\frac{b}{2}\right)\times\frac{1}{\sqrt{b^2-\left(\frac{b}{2}\right)^2}}$ ✓

$g'\left(\frac{b}{2}\right) = 2\times sin^{-1}\frac{1}{2}\times\frac{1}{\sqrt{b^2-\frac{b^2}{4}}}$

$g'\left(\frac{b}{2}\right) = 2\times\frac{\pi}{6}\times\frac{1}{\sqrt{\frac{3b^2}{4}}}$

$g'\left(\frac{b}{2}\right) = \frac{\pi}{3}\times\frac{1}{\frac{b}{2}\sqrt{3}} = \frac{\pi}{3}\times\frac{2}{b\sqrt{3}}$

We are given that $g'\left(\frac{b}{2}\right) = \frac{2\pi}{3}$ this means

$\frac{\pi}{3}\times\frac{2}{b\sqrt{3}} = \frac{2\pi}{3}$

Dividing on both sides by $\frac{2\pi}{3}$, we get

$\frac{1}{b\sqrt{3}} = 1$ that is $b\sqrt{3} = 1$ Hence, $b = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. ✓

d) The groups of dancers formed will be combinations as the order of the members in each group is not

important.
So, the number of 2 person groups will be
$\binom{n}{2} = \frac{n!}{2!\,(n-2)!}$
Also, the number of 3 person groups will be
$\binom{n}{3} = \frac{n!}{3!\,(n-3)!}$
Now, as the number of different groups of three dancers he can select is 3 times the number of different groups of two dancers then
$\frac{n!}{3!\,(n-3)!} = 3 \times \frac{n!}{2!\,(n-2)!}$ ✓
Dividing both sides by $n!$, we get
$\frac{1}{3!\,(n-3)!} = \frac{3}{2!\,(n-2)!}$
Multiplying both sides by $3!\,(n-2)!$, we get
$n - 2 = 9$ So, $n = 11$. ✓
Hence, the number of dancers in this academy is 11.

e) Consider $9^{n+3} - 2^{2n+2}$.
For $n = 1$, the statement is $9^4 - 2^4 = 6\,545$ ✓
which is divisible by 5.
Hence, the statement is true for $n = 1$.
Assume the statement is true for $n = k$ that is
$9^{k+3} - 2^{2k+2} = 5p$, where p is a positive integer.
For $n = k + 1$ the statement is
$9^{k+4} - 2^{2k+4} = 9 \times 9^{k+3} - 2^{2k+4}$
By substituting $9^{k+3} = 5p + 2^{2k+2}$ from assumption, we get
$9^{k+4} - 2^{2k+4}$
$= 9 \times (5p + 2^{2k+2}) - 2^2 \times 2^{2k+2}$ ✓
$= 45p + 9 \times 2^{2k+2} - 4 \times 2^{2k+2}$
$= 45p + 5 \times 2^{2k+2} = 5 \times (9p + 2^{2k+2})$ ✓
Let $M = 9p + 2^{2k+2}$, Where M is a positive integer
then $9^{k+4} - 2^{2k+4} = 5M$ which is clearly divisible by 5.
Hence, if the statement is true for $n = k$ it is also true for $n = k + 1$.
The statement was proved true for n = 1 and by mathematical induction it is true for n = 2, n = 3 and so on. Hence, it is true for all values of n ≥ 1.

QUESTION 13

a) i) From the diagram we can see that the period of the cosine curve is π radians.
Given the period of a cosine curve is $T = \frac{2\pi}{b}$ this means $\frac{2\pi}{b} = \pi$, So $b = 2$. ✓
ii) As the equation of the curve is $x = a + \cos 2y$ then the volume of the solid formed is

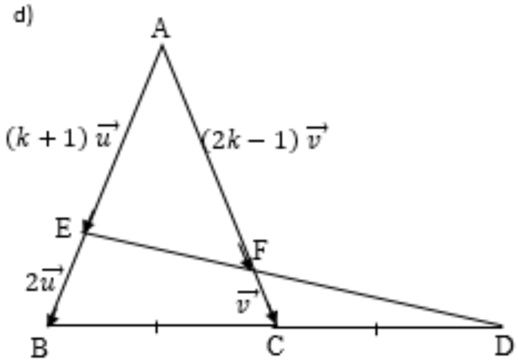
$$V = \pi \int_0^{2\pi} (a + \cos 2y)^2\, dy$$
$$V = \pi \int_0^{2\pi} a^2 + 2a\cos 2y + \cos^2 2y \quad dy$$
Note: $\cos^2 2y = \frac{1}{2}(\cos 4y + 1) = \frac{1}{2}\cos 4y + \frac{1}{2}$
$$V = \pi \int_0^{2\pi} a^2 + 2a\cos 2y + \frac{1}{2}\cos 4y + \frac{1}{2} \; dy$$ ✓
$$V = \pi \left[a^2 y + a\sin 2y + \frac{1}{8}\sin 4y + \frac{y}{2} \right]_0^{2\pi}$$
$$V = \pi \left[\left(2\pi a^2 + a\sin 4\pi + \frac{1}{8}\sin 8\pi + \pi \right) - (0) \right]$$
Note: $\sin 4\pi = \sin 8\pi = 0$ this means
$V = \pi\,(2\pi a^2 + \pi)$ that is
$V = \pi^2(2a^2 + 1)$
So, $\pi^2(2a^2 + 1) = 19\pi^2$
Dividing by π^2 on both sides, we get
$2a^2 + 1 = 19$ that is $2a^2 = 18$
So, $a^2 = 9$ this means $a = \pm 3$. Now, as the given curve is to the right of the y axis then $a = 3$. ✓

b) Assuming the product manager's claim is correct.
Let X be the number of juice bottles which contains less than 375 mL.
X follows a binomial distribution Bin (18, 0.25),
$\mu = np = 18 \times 0.25 = 4.5$
$\sigma(X) = \sqrt{npq} = \sqrt{18 \times 0.25 \times 0.75} = \sqrt{3.375}$
If we approximate X by a normal distribution without the continuity correction, we get
$$P(X) \geq 9 = P\left(z \geq \frac{9 - 4.5}{\sqrt{3.375}}\right) = P(z \geq 2.45)$$ ✓
Now, by using the table in the question booklet, we get $P(z \leq 2.45) = 0.9929$.
this means
$P(z \geq 2.45) = 1 - 0.9929 = 0.0071$
Hence, $P = 0.71\%$.
Note: If we approximate X by a normal distribution with the continuity correction, we get ✓ either P=0.71% or P=1.46%
$$P(X) \geq 9 = P\left(z \geq \frac{8.5 - 4.5}{\sqrt{3.375}}\right) = P(z \geq 2.18)$$
$P(z \leq 2.18) = 0.9854$ this means
$P(z \geq 2.18) = 1 - 0.9854 = 0.0146$
Hence, $P = 1.46\%$.
The normal distribution will be a good approximation of the binomial distribution if both $np > 5$ and $n(1-p) > 5$.
In our question $np = 18 \times 0.25 = 4.5 < 5$ ✓

and $n(1-p) = 18 \times 0.75 = 13.5$.
As only $n(1-p) > 5$ then the normal approximation may not be valid.

c) Let $g(x)$ the monic polynomial with degree 3 and roots α, β and γ be $g(x) = x^3 + bx^2 + cx + d$.
So, $g'(x) = 3x^2 + 2bx + c$
and $g''(x) = 6x + 2b$ this means ✓
$g''(\alpha + \beta + \gamma) = 6(\alpha + \beta + \gamma) + 2b = 20$
But as α, β and γ are the roots of $g(x)$, then
$\alpha + \beta + \gamma = -b$ this indicates that
$6 \times -b + 2b = 20$ that is $-4b = 20$. Hence, $b = -5$. ✓
Therefore, $\alpha + \beta + \gamma = -b = 5$ (1)
Also, $g''(\alpha^2 + \beta^2 + \gamma^2) = 6(\alpha^2 + \beta^2 + \gamma^2) + 2b$
This means $6(\alpha^2 + \beta^2 + \gamma^2) + 2b = 164$ that is
$6(\alpha^2 + \beta^2 + \gamma^2) - 10 = 164$
$So, \alpha^2 + \beta^2 + \gamma^2 = 29$ (2)
Now, $\alpha(\beta + \gamma) + \beta(\alpha + \gamma) + \gamma(\alpha + \beta)$
$= \alpha(\beta + \gamma + \alpha - \alpha) + \beta(\alpha + \gamma + \beta - \beta) + \gamma(\alpha + \beta + \gamma - \gamma)$
By substituting $\alpha + \beta + \gamma = 5$, we get
$= 5\alpha - \alpha^2 + 5\beta - \beta^2 + 5\gamma - \gamma^2$
$= 5(\alpha + \beta + \gamma) - (\alpha^2 + \beta^2 + \gamma^2)$
By substituting (1) and (2), we get
$\alpha(\beta + \gamma) + \beta(\alpha + \gamma) + \gamma(\alpha + \beta)$
$= 5(5) - (29) = -4$ ✓

d)



$\overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AF} = -(k+1)\vec{u} + (2k-1)\vec{v}$ ✓
$\overrightarrow{ED} = \overrightarrow{EB} + \overrightarrow{BD}$ but $\overrightarrow{BC} = \overrightarrow{CD}$ (given) this means
$\overrightarrow{ED} = \overrightarrow{EB} + 2\overrightarrow{BC}$
Also, $\overrightarrow{BC} = \overrightarrow{BE} + \overrightarrow{EF} + \overrightarrow{FC}$ this means
$\overrightarrow{ED} = \overrightarrow{EB} + 2\left(\overrightarrow{BE} + \overrightarrow{EF} + \overrightarrow{FC}\right)$
$\overrightarrow{ED} = 2\vec{u} + 2[-2\vec{u} - (k+1)\vec{u} + (2k-1)\vec{v} + \vec{v}]$
$\overrightarrow{ED} = 2\vec{u} + 2[-2\vec{u} - k\vec{u} - \vec{u} + 2k\vec{v} - \vec{v} + \vec{v}]$
$\overrightarrow{ED} = 2\vec{u} + 2[-3\vec{u} - k\vec{u} + 2k\vec{v}]$
$\overrightarrow{ED} = 2\vec{u} - 6\vec{u} - 2k\vec{u} + 4k\vec{v}$
$\overrightarrow{ED} = -4\vec{u} - 2k\vec{u} + 4k\vec{v}$
$\overrightarrow{ED} = (-4 - 2k)\vec{u} + 4k\vec{v}$ ✓
Now, we are given that $\overrightarrow{EF} = p\,\overrightarrow{ED}$ this means
$-(k+1)\vec{u} + (2k-1)\vec{v} = p\,[(-4-2k)\vec{u} + 4k\vec{v}]$
$-(k+1)\vec{u} + (2k-1)\vec{v} = p(-4-2k)\vec{u} + 4pk\vec{v}$
This indicates that
$-(k+1) = p(-4-2k)$ and $2k - 1 = 4pk$
$(k+1) = p(4+2k)$ and $2k - 1 = 4pk$
$\frac{k+1}{4+2k} = p$ and $\frac{2k-1}{4k} = p$
Therefore, $\frac{k+1}{4+2k} = \frac{2k-1}{4k}$
$4k^2 + 4k = 8k - 4 + 4k^2 - 2k$
$-2k = -4$ So, $k = 2$
Hence, $p = \frac{2+1}{4+2\times 2} = \frac{3}{8}$. ✓

e) $f(x)$ will have an inverse function $f^{-1}(x)$ in the domain $x \geq e$ if is one to one function. ✓ This means $f'(x)$ must be strictly positive or strictly negative.
Now, we need to study the behaviour of $f'(x) = x\ln x - x + 1$ by finding $f''(x)$.
By using the product rule, we get
$u = x$ and $v = \ln x$
$u' = 1$ $v' = \frac{1}{x}$
So $f''(x) = \ln x + x \times \frac{1}{x} - 1 = \ln x$. ✓
Now, in the domain $x \geq e$, the curve $y = \ln x$ is an increasing curve starting at the point $(e, 1)$ which means $f''(x)$ is always positive. Also, this indicates that $f'(x)$ is always increasing and as $f'(1) = 0$ this means $f'(x)$ starts from zero and after that is always positive. Therefore, $f(x)$ is always increasing in the domain $x \geq e$. Hence, $f(x)$ is one to one function and has an inverse function in the domain $x \geq e$. ✓

QUESTION 14

a) $(e^x + 1)\frac{dy}{dx} = e^x \sec y$
$(e^x + 1)\,dy = \frac{e^x}{\cos y}dx$
$\cos y\ dy = \frac{e^x}{e^x + 1}\ dx$ ✓
$\int \cos y\ dy = \int \frac{e^x}{e^x + 1}\ dx$
$\sin y = \ln|e^x + 1| + c$ ✓
Now, $e^x + 1 > 0$ therefore $|e^x + 1| = e^x + 1$.

So, $\sin y = \ln(e^x + 1) + c$

As the graph of this equation passes through the origin this indicates that when $x = 0,\ y = 0$.

$\sin 0 = \ln(e^0 + 1) + c$

$0 = \ln 2 + c$ that is $c = -\ln 2$. ✓

Therefore, $\sin y = \ln(e^x + 1) - \ln 2$ this means

$\sin y = \ln\left(\frac{e^x + 1}{2}\right)$

$y = \sin^{-1}\left(\ln\left(\frac{e^x+1}{2}\right)\right)$.

b) i) At time t, the height reached by the water in tank Q is h that indicates the volume of water that passed from tank P to tank Q was

$V_Q = a\sqrt{2} \times b\sqrt{2} \times h = 2abh$.

Let the decrease in the volume in cylinder P be

$V_P = abx$, where x is the drop in height.

As $V_P = V_Q$ then $abx = 2abh$. So, $x = 2h$.

Therefore, when the height of water in cylinder Q reached h, the height of water in cylinder P was $H - x = H - 2h$. ✓

Hence, the rate of change is

✓ $\frac{dh}{dt} = k[(H - 2h) - h] = k(H - 3h)$

Alternative method

The area of the base of tank P is

$a \times b = ab\ cm^2$

The area of the base of tank Q is

$a\sqrt{2} \times b\sqrt{2}\ cm^2 = 2ab\ cm^2$

Hence, for every 2 cm decrease in tank P there will be a 1 cm increase in tank Q.

This can be shown in the following table.

Height in tank P is H	H	$H-2$	$H-4$	$H-6$
Height in tank Q is h	0	1	2	3
Difference in heights is $H-h$	$H-0$	$H-3$	$H-6$	$H-9$

By looking at the last row, which lists the difference between the two tanks, it can be seen that the expressions can be generalised as $H - 3h$.

The change in h can be expressed as $\frac{dh}{dt}$.

As $\frac{dh}{dt}$ is proportional to the difference in heights,

$\frac{dh}{dt} \propto H - 3h$ so $\frac{dh}{dt} = k(H - 3h)$.

ii) $\frac{dh}{dt} = k(H - 3h)$ this means

$\int \frac{1}{H - 3h}\, dh = \int k\ dt$

$-\frac{1}{3}\int \frac{-3}{H - 3h}\, dh = \int k\ dt$

$\int \frac{-3}{H - 3h}\, dh = -3\int k\ dt$

} ✓ either

$\ln|H - 3h| = -3kt + c$

Now, $H - 3h > 0$ (as the liquid will flow from tank P to tank Q if the difference in height is positive)

then $|H - 3h| = H - 3h$.

So, $\ln(H - 3h) = -3kt + c$

When $t = 0, h = 0$ so $\ln H = c$ this means

$\ln(H - 3h) = -3kt + \ln H$

$\ln(H - 3h) - \ln H = -3kt$

$\ln\left(\frac{H-3h}{H}\right) = -3kt$ this means $\frac{H-3h}{H} = e^{-3kt}$

$H - 3h = He^{-3kt}t$

$3h = H - He^{-3kt}t$

$3h = H(1 - e^{-3kt})$

Hence, $h = \frac{H}{3}(1 - e^{-3kt})$ ✓ correct progress and justification to this

iii) When $t = 4,\ h_1 = \frac{H}{3}(1 - e^{-12k})$.

When $t = 12,\ h_2 = \frac{H}{3}(1 - e^{-36k})$.

As $h_2 = \frac{7}{4}h_1$ this means

$\frac{H}{3}(1 - e^{-36k}) = \frac{7}{4} \times \frac{H}{3}(1 - e^{-12k})$ ✓

Dividing both sides by $\frac{H}{3}$, we get

$1 - e^{-36k} = \frac{7}{4}(1 - e^{-12k})$

$4 - 4e^{-36k} = 7 - 7e^{-12k}$

$7e^{-12k} - 4e^{-36k} - 3 = 0$ ✓ or equivalent

Let $m = e^{-12k}$ this means $m^3 = e^{-36k}$ and the equation becomes $7m - 4m^3 - 3 = 0$

Multiplying by -1, we get

$4m^3 - 7m + 3 = 0$

Using the factor theorem $m = 1$ is a solution, so $(m - 1)$ is a factor.

We can find the other factors by division.

$$
\begin{array}{r l}
 & 4m^2 + 4m - 3 \\
m - 1 \,\big) & 4m^3 + 0m^2 - 7m + 3 \\
 & \underline{-4m^3 + 4m^2} \\
 & \qquad 4m^2 - 7m + 3 \\
 & \qquad \underline{-4m^2 + 4m} \\
 & \qquad\qquad -3m + 3 \\
 & \qquad\qquad \underline{3m - 3} \\
 & \qquad\qquad\quad 0
\end{array}
$$

So, $4m^3 - 7m + 3 = 0$

$(m-1)(4m^2 + 4m - 3) = 0$

$m = 1$ or $4m^2 + 4m - 3 = 0$

$(2m-1)(2m+3) = 0$

$m = \frac{1}{2}$ or $m = -\frac{3}{2}$

As $m = e^{-12k}$ and exponential is always positive then $m = -\frac{3}{2}$ is invalid.

Also, as $h_1 = \frac{H}{3}\left(1 - e^{-12k}\right)$ then $m = 1$ means $e^{-12k} = 1$ which means $h_1 = 0$ which is also invalid. This means $e^{-12k} = \frac{1}{2}$. ✓

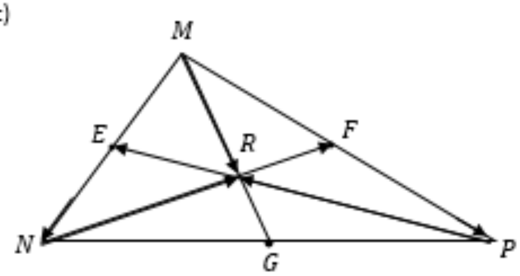
Now, when $t = 16$,

$h = \frac{H}{3}\left(1 - e^{-48k}\right) = \frac{H}{3}\left(1 - \left(e^{-12k}\right)^4\right)$.

By substituting $e^{-12k} = \frac{1}{2}$, we get

$h = \frac{H}{3}\left(1 - \left(\frac{1}{2}\right)^4\right) = \frac{H}{3} \times \frac{15}{16} = \frac{5H}{16}$. ✓

c)



$\overrightarrow{NF} = \overrightarrow{NM} + \overrightarrow{MF}$

$= \overrightarrow{NM} + \frac{1}{2}\overrightarrow{MP}$ (as F is the midpoint of MP)

$= -\underset{\sim}{n} + \frac{1}{2}\underset{\sim}{p}$

✓ either vector NF or NR in terms of vectors n and p

$\overrightarrow{NR} = x\,\overrightarrow{NF} = x\left(-\underset{\sim}{n} + \frac{1}{2}\underset{\sim}{p}\right)$

$= -x\underset{\sim}{n} + \frac{1}{2}x\underset{\sim}{p}$

Now, $\overrightarrow{MR} = \overrightarrow{MN} + \overrightarrow{NR}$

$= \underset{\sim}{n} - x\underset{\sim}{n} + \frac{1}{2}x\underset{\sim}{p} = (1-x)\,\underset{\sim}{n} + \frac{x}{2}\underset{\sim}{p}$ (1)

Also, $\overrightarrow{PE} = \overrightarrow{PM} + \overrightarrow{ME}$

$= \overrightarrow{PM} + \frac{1}{2}\overrightarrow{MN}$ (as E is the midpoint of MN)

$= -\underset{\sim}{p} + \frac{1}{2}\underset{\sim}{n}$

$\overrightarrow{PR} = y\,\overrightarrow{PE} = y\left(-\underset{\sim}{p} + \frac{1}{2}\underset{\sim}{n}\right)$

$= -y\underset{\sim}{p} + \frac{1}{2}y\underset{\sim}{n}$

$\overrightarrow{MR} = \overrightarrow{MP} + \overrightarrow{PR}$

$\overrightarrow{MR} = \underset{\sim}{p} + -y\underset{\sim}{p} + \frac{1}{2}y\underset{\sim}{n} = (1-y)\underset{\sim}{p} + \frac{y}{2}\underset{\sim}{n}$ (2) ✓ two expressions for vector MR

By equating (1) and (2), we get

$(1-x)\underset{\sim}{n} + \frac{x}{2}\underset{\sim}{p} = (1-y)\underset{\sim}{p} + \frac{y}{2}\underset{\sim}{n}$

By equating the components of the vectors, we get

$1 - x = \frac{y}{2}$ that is $2 - 2x = y$ (3)

and $1 - y = \frac{x}{2}$ that is $2 - 2y = x$ (4)

Now, by substituting (3) into (4), we get

$2 - 2(2 - 2x) = x$ that is

$2 - 4 + 4x = x$ So, $x = \frac{2}{3}$.

By substituting $x = \frac{2}{3}$ in (4), *we get*

$2 - 2y = \frac{2}{3}$ that is $6 - 6y = 2$

So, $-6y = -4$

Therefore $y = \frac{2}{3}$. ✓ and x = 2/3

Now, $\overrightarrow{NR} = \frac{2}{3}\,\overrightarrow{NF}$ this means $\overrightarrow{RF} = \frac{1}{3}\,\overrightarrow{NF}$.

Hence, the ratio of

$\overrightarrow{NR} : \overrightarrow{NF} = \frac{2}{3} : \frac{1}{3} = 2 : 1$ (A)

Also, $\overrightarrow{PR} = \frac{2}{3}\,\overrightarrow{PE}$ this means $\overrightarrow{RE} = \frac{1}{3}\,\overrightarrow{PE}$

Hence, the ratio of

$\overrightarrow{PR} : \overrightarrow{RE} = \frac{2}{3} : \frac{1}{3} = 2 : 1$ (B)

By substituting $y = \frac{2}{3}$ in (2),we get

$\overrightarrow{MR} = \left(1 - \frac{2}{3}\right)\underset{\sim}{p} + \left(\frac{1}{2} \times \frac{2}{3}\right)\underset{\sim}{n}$

$\overrightarrow{MR} = \frac{1}{3}\,\underset{\sim}{p} + \frac{1}{3}\underset{\sim}{n} = \frac{1}{3}(\underset{\sim}{p} + \underset{\sim}{n})$

Now, $\overrightarrow{MG} = \overrightarrow{MN} + \overrightarrow{NG}$

$= \overrightarrow{MN} + \frac{1}{2}\overrightarrow{NP}$ (as G is the midpoint of PN)

This means $\overrightarrow{MG} = \underset{\sim}{n} + \frac{1}{2}(-\underset{\sim}{n} + \underset{\sim}{p})$

$= \underset{\sim}{n} - \frac{1}{2}\underset{\sim}{n} + \frac{1}{2}\underset{\sim}{p}$

$= \frac{1}{2}\underset{\sim}{n} + \frac{1}{2}\underset{\sim}{p} = \frac{1}{2}(\underset{\sim}{n} + \underset{\sim}{p})$

Also, $\overrightarrow{RG} = \overrightarrow{MG} - \overrightarrow{MR}$

$= \overrightarrow{MG} - \overrightarrow{MR}$

$= \frac{1}{2}(\underset{\sim}{n} + \underset{\sim}{p}) - \frac{1}{3}(\underset{\sim}{n} + \underset{\sim}{p}) = \frac{1}{6}(\underset{\sim}{n} + \underset{\sim}{p})$

So, the ratio of

$\overrightarrow{MR} : \overrightarrow{RG} = \frac{1}{3}(\underset{\sim}{p} + \underset{\sim}{n}) : \frac{1}{6}(\underset{\sim}{p} + \underset{\sim}{n})$

$= \frac{1}{3} : \frac{1}{6} = 2 : 1$ (C)

Hence, from (A) , (B) and (C) we can deduce that the three medians are concurrent at R and the point *R* divides each of these medians in the ratio 2 : 1. ✓